

Anchored expansion and random walk

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Abstract

This paper studies anchored expansion, a non-uniform version of the strong isoperimetric inequality. We show that every graph with i -anchored expansion contains a subgraph with isoperimetric (Cheeger) constant at least i . We prove a conjecture by Benjamini, Lyons and Schramm (1999) that in such graphs the random walk escapes with a positive lim inf speed. We also show that anchored expansion implies a heat-kernel decay bound of order $\exp(-cn^{1/3})$.

1 Introduction

Anchored expansion was introduced by Benjamini, Lyons, and Schramm (1999) as a non-uniform version of the strong isoperimetric inequality, after Thomassen (1992) used more general “anchored” isoperimetric inequalities to give sufficient conditions for the transience of random walks on graphs. Let $G = (V, w)$ be an infinite weighted graph, that is a countable set V together with a symmetric, nonnegative function $w((u, v))$, and let E denote the set of edges, that is unordered pairs (u, v) where w is positive. Define the **weight** $w(v)$ of a vertex v as the sum of the weights over the incident edges; we will assume that this is finite for every vertex. Define the **volume** $|\cdot|$ of an edge or vertex set as the sum of the weights over the set, and define the **edge boundary** ∂S of a vertex set S as the set of edges with one vertex inside S and one outside. The **strong isoperimetric inequality** with constant i , perhaps the simplest isoperimetric inequality, states that

$$|\partial S| \geq i|S| \tag{1.1}$$

for all vertex sets S , and the **Cheeger constant** of a graph is the supremum of the i for which this inequality holds. The implications of a positive Cheeger constant are very strong; the Markov kernel on a graph with positive Cheeger constant has spectral radius less than 1 (Cheeger (1970), Dodziuk (1984), Mohar (1988); these two conditions are in fact equivalent) and as a result, if the graph does not grow faster than exponentially, the random

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walk escapes at a positive speed, (i.e. linear rate). As is clear from the definition, having a positive Cheeger constant is a rather fragile property. Random perturbations, such as Bernoulli percolation on an (unweighted) graph, even with a very high retention probability, or a geometric stretching of edges, destroy it; see BLS (1999) for results on stability of graph properties under random perturbations.

A more stable condition, which we call ***i*-anchored expansion**, requires (1.1) to hold with possibly some exceptions provided that every vertex is contained in only finitely many connected exceptional S (of course, there need not be a uniform bound on the number of such exceptions). We call the supremum of i for which a graph G has i -anchored expansion the **anchored expansion constant** $\mathbf{i}(G)$; if this constant is greater than 0, then we say that G has the **anchored expansion property**. If G is connected with edge weights bounded from below (in particular, if G is a connected, unweighted graph), then this is equivalent to a version of the original definition,

$$\mathbf{i}(G) = \liminf_{S \ni v} \frac{|\partial S|}{|S|},$$

here S ranges over all connected vertex sets containing a fixed vertex v ; the \liminf is then applied to the set of values obtained. This definition does not depend on the choice of the fixed vertex, and it explains the name “anchored expansion”. The only difference from the original definition (given for the unweighted case) is that here the volume of a vertex set is defined as the sum of degrees rather than the number of vertices.

Benjamini, Lyons and Schramm (1999) conjectured that in an unweighted graph with bounded degree and anchored expansion, the random walk has positive \liminf speed with positive probability. The main goal of this paper is to prove this conjecture in a slightly stronger form. Say a weighted graph has w_0 -**bounded geometry** if all positive edge weights are at least 1 and all vertex weights are at most w_0 , and let $|X_n|$ denote the graph distance between the random walker and the initial vertex at time n .

Theorem 1.1 *There exists $c > 0$ so that the random walk on a weighted graph with w_0 -bounded geometry satisfies $\liminf |X_n|/n \geq c\mathbf{i}(G)^7 w_0^{-3}$ a.s.*

This theorem gives a geometric explanation for positive speed in certain graphs, such as infinite components of p -Bernoulli percolation on graphs with positive Cheeger constant for high p , geometric edge-stretchings of such graphs, or supercritical Galton-Watson trees. For these graphs, Chen and Peres (1998), inspired by questions of Benjamini, Lyons, and Schramm (1999), proved the anchored expansion property. It is an open question (see Häggström, Schonmann and Steif (1999)) whether infinite clusters of Bernoulli percolation on a transitive graph have the anchored expansion property. The same authors prove that if G is a transitive graph and there exists an automorphism-invariant percolation on G where all infinite components have the anchored expansion property, then G has positive Cheeger constant.

Since an exponential heat kernel bound is equivalent to having a positive Cheeger constant, one cannot hope that anchored expansion would imply such a strong bound. The following theorem gives a sub-exponential bound, which is strongest in the sense that the $n^{1/3}$ in the exponent cannot be improved.

Theorem 1.2 *Let G be a weighted graph with the anchored expansion property and w_0 -bounded geometry. Let $\alpha := \mathbf{i}(G)^2(w_0^2/2)^{-1/3}/9$. For every vertex x there is an N so that*

$$p^n(x, y) < e^{-\alpha n^{1/3}} \quad \text{for all } n > N, \ y \in V.$$

Varopoulos (1991) showed that such a bound holds in Cayley graphs of any finitely generated group of exponential growth (see Hebisch and Saloff-Coste (1993) for a more general statement and a simpler proof). Such a group can be amenable (e.g. the lamplighter group G_1), in which case it is an example where such decay holds but anchored expansion does not.

Section 3 examines the geometry of graphs with i -anchored expansion, and shows that they are built from a graph with Cheeger constant at least i and “islands”, each of which is finite but whose size is not necessarily bounded by a constant (a binary tree with “pipes” of increasing length attached to a scarce set of vertices is a graph with anchored expansion and unbounded “islands”). Section 4 proves some properties of random walk on such graphs. Using these results, in Section 5 we prove that the random walk has positive speed, and in Section 6 we establish the heat kernel bound.

2 Notation

The concept of **the random walk on a weighted graph** $G = (V, w)$ is just a geometric representation of a countable, reversible Markov chain with transition probabilities

$$p(u, v) = w((u, v))/w(u).$$

Conversely, if we have a reversible Markov chain with transition probabilities $p(u, v)$ and stationary measure $w(v)$ we get a weighted graph by the above formula. We will use the notation P for the transition probability matrix of the walk. We will usually denote the walk itself by $\{X_n\}$, and \mathbf{P}_v , \mathbf{E}_v will mean probability and expectation with respect to the walk started at vertex v .

We will consider the Hilbert space $L^2(V, w)$ of functions, equipped with the inner product and norm

$$(f, g) = \sum_{v \in V} f(v)g(v)w(v), \quad \|f\| = (f, f)^{\frac{1}{2}}.$$

For an operator P on $L^2(V, w)$, we will use the norm $\|P\| = \sup \|Pf\|/\|f\|$. The **Markov kernel** P of a random walk on a weighted graph is the operator on $L^2(V, w)$ defined by $(Pf)(v) := \mathbf{E}_v f(X_1)$.

We will use the standard notation δ_x , for unit mass at x (formally, δ_x is an element of the dual of $L^2(V, w)$), and $\mathbf{1}_S$ for the indicator (characteristic) function of the set S . The volume of a vertex set S can be written as $|S| = \sum_{v \in S} w(v) = \|\mathbf{1}_S\|^2$. The **number of vertices** of S will be denoted $\#S$.

The **induced subgraph** of $S \subset V$ is the graph G with vertex set S and edge set given by $\{(u, v) \in E : u, v \in S\}$, and we call S **connected** if its induced subgraph is connected. We will often write $G \setminus S$ for the induced subgraph of $V \setminus S$. The **inner vertex boundary**

of S is the set of vertices in S with a neighbor outside S , and the **outer vertex boundary** of S the set of vertices outside S with a neighbor in S .

By a **path of length n** we mean a subgraph with vertex set (v_0, \dots, v_n) and edge set consisting of edges between consecutive v -s. The **graph distance** between two vertices in G is the length of the shortest path with endpoints given by the two vertices. The notation $|v|$ for a vertex denotes the graph distance between v and some declared fixed vertex; this vertex will usually be the starting point of the random walk we study. The **lim inf speed** of the walk is defined as $\liminf |X_n|/n$.

We will use c, c_2, c_3 to denote constants whose values might change from one expression to another.

3 Geometry of graphs with anchored expansion

Let $G = (V, w)$ be a weighted graph. For $i < 1$, define the **i -isolation** of a finite-volume vertex set S of G by

$$\Delta_i S = i|S| - |\partial S|.$$

A vertex set S with positive i -isolation will be called **i -isolated**. We will call a vertex set S satisfying $\Delta_i S > \Delta_i A$ for every subset $A \neq S$ of S an **i -isolated core**. Since A can be the empty set, an isolated core must be either empty or isolated. A nice property of i -isolated cores is given in the following lemma.

Lemma 3.1 *Let A be a vertex set and let S be an i -isolated core which is not a subset of A . Then $\Delta_i(A \cup S) > \Delta_i A$.*

PROOF. Note that if B and C are disjoint vertex sets then

$$\Delta_i(B \cup C) = \Delta_i B + \Delta_i C + 2|\partial B \cap \partial C|. \quad (3.2)$$

The factor 2 in the above expression appears since common boundary edges of B and C are not boundary edges of their union. Then $\Delta_i(A \cup S) = \Delta_i(A \setminus S) + \Delta_i S + 2|\partial(A \setminus S) \cap \partial S|$. The hypothesis can be used to bound the second term. The last term equals twice the total weight on edges with one endpoint in $A \setminus S$ and the other endpoint in S ; this does not increase if we require the latter endpoint to be in a subset of S . Therefore

$$\Delta_i(A \cup S) > \Delta_i(A \setminus S) + \Delta_i(A \cap S) + 2|\partial(A \setminus S) \cap \partial(A \cap S)| = \Delta_i A. \quad \square$$

Corollary 3.2 *The union of finitely many i -isolated cores is an i -isolated core.*

PROOF. It suffices to prove this for two i -isolated cores S, S' . Let $A \subset S \cup S'$. Two applications of the lemma imply $\Delta_i A \leq \Delta_i(A \cup S) \leq \Delta_i(A \cup S \cup S')$, and one inequality is strict unless $A = S \cup S'$. \square

Let A_i denote the union of all i -isolated cores in G . It follows from the definitions and Corollary 3.2 that if G has i -anchored expansion then every connected component of A_i is a *finite* union of isolated cores, hence an i -isolated core.

If G has i -anchored expansion, then the set A_i has the remarkable property that $G \setminus A_i$ is a graph with Cheeger constant at least i . Indeed, let S be a finite subset of $V \setminus A_i$, and let C be a (possibly empty) i -isolated core containing all vertices adjacent to S in A_i . A (possibly empty) minimal subset B of $C \cup S$ satisfying $\Delta_i B \geq \Delta_i(C \cup S)$ must be an i -isolated core, hence $B \subset A_i$, and thus $B \subseteq C$. Since C is an i -isolated core, we get

$$\Delta_i(C \cup S) \leq \Delta_i B \leq \Delta_i C. \quad (3.3)$$

Let $\Delta_i^{G \setminus A_i}$ denote i -isolation of vertex sets in the graph $G \setminus A_i$. When A_i is removed from G , the volumes of both S and ∂S decrease by $|\partial C \cap \partial S|$. Thus we get $\Delta_i^{G \setminus A_i} S = \Delta_i S + (1 - i)|\partial C \cap \partial S|$. Expressing $\Delta_i S$ by (3.2) gives

$$\Delta_i^{G \setminus A_i} S = \Delta_i(C \cup S) - \Delta_i C - 2|\partial C \cap \partial S| + (1 - i)|\partial C \cap \partial S|.$$

This is at most 0 by (3.3), and we get the required isoperimetric inequality. Thus we have shown

Proposition 3.3 *Every graph with i -anchored expansion contains a subgraph with Cheeger constant at least i .*

Note that $G \setminus A_i$ is an isomorphism-invariant function of the graph G . If, for example, G' is a transitive graph and $G \subset G'$ is a random subgraph whose law is invariant under a group of automorphisms of G' , then the law of $G \setminus A_i$ is also invariant under this group.

We will call the connected components of A_i (i -)islands, and $G \setminus A_i$ the oceans (plural since $G \setminus A_i$ is not always connected). If $i' < i$, then we have $\Delta_i S = (i - i')|S| + \Delta_{i'} S$, so i' -isolated sets are also i -isolated, and if $A \subset S$ and $\Delta_{i'} A < \Delta_{i'} S$, then $\Delta_i A < \Delta_i S$. In particular, i' -isolated cores are i -isolated cores as well, giving $A_{i'} \subset A_i$. Thus decreasing i has the effect of global warming: it raises the level of the oceans. The following lemma gives an upper bound on how much the level needs to be raised to sink certain islands.

Lemma 3.4 *Let G be a connected graph with i -anchored expansion and edge weights bounded below by 1. Let S be a union of islands, each having volume at most i'^{-1} for some $i' > 0$. Then $S \subset V \setminus A_{i'}$.*

PROOF. Any i -island in S has boundary volume at least 1 and positive i -isolation, thus volume greater than i^{-1} . This gives $i' < i$. Similarly, each i' -island has volume greater than i'^{-1} , so no i' -island is a subset of an island in S . But since $A_{i'} \subset A_i$, this implies that $A_{i'} \cap S = \emptyset$. \square

Let N_m denote the m th positive time the walk is in $G \setminus A_i$. The process $\{X_{N_m}\}$ (often called the induced Markov chain on $G \setminus A_i$) is also a reversible Markov chain, that is a random walk on a weighted graph G_i . If G is connected, then so is G_i . The vertex set of G_i is given by $V \setminus A_i$, and its edge weight function satisfies

$$w_i((u, v)) = w(u) \mathbf{P}_u(X_{N_1} = v).$$

Clearly, for $u, v \in V \setminus A_i$ we have $w_i((u, v)) = w((u, v))$ unless u and v are both on the outer vertex boundary of the same island in G . It is also clear that for $v \in V(G_i)$, we have $w_i(v) = w(v)$. The reversibility of the walk on G implies that w_i is a symmetric function on the edges.

The graph G_i has the same vertex set as $G \setminus A_i$, but its edge and vertex weights are greater or equal. We now show that G_i also has Cheeger constant at least i . To see this, let S be a finite subset of $V \setminus A_i$ and follow the argument for $G \setminus A_i$ to get (3.3). Let superscript G_i on volume or i -isolation denote these quantities measured in the base graph G_i . Since we have $|S|^{G_i} = |S|$ and $|\partial S|^{G_i} \geq |\partial S| - |\partial C \cap \partial S|$, it follows by (3.2) and (3.3) that

$$\Delta_i^{G_i} S \leq \Delta_i S + |\partial C \cap \partial S| = \Delta_i(C \cup S) - \Delta_i C - |\partial C \cap \partial S| \leq 0.$$

The upcoming analysis of random walks will need a rigorous formulation of the idea that large islands cannot be very close to each other. One could expect islands to have a neighborhood, whose radius depends on the size of the island, within which there are no other islands; or if this cannot be achieved, then at least one could group nearby islands together to get such a configuration. This is too optimistic as said, but Proposition 3.6 has a similar decomposition, for which we first have to introduce some tools.

A **bridge structure** interconnecting a vertex set $S \subset V(G)$ is a set of vertices B so that $B \cup S$ is a connected vertex set. A **bridge** connecting two vertex sets $S_1, S_2 \subset V(G)$ is a vertex set B so that the vertex set $B \cup S_1 \cup S_2$ has a connected component intersecting both S_1 and S_2 . Define the **i -length** of a bridge B as the number of its vertices in the ocean, $\#(B \setminus A_i)$. For a vertex set S and a vertex $v \notin S$ let $\text{dist}_i(v, S)$ equal 1 plus the i -length of the shortest bridge connecting $\{v\}$ and S ; for $v \in S$, let $\text{dist}_i(v, S) := 0$.

Lemma 3.5 *Let G be a weighted graph with w_0 -bounded geometry and $\mathbf{i}(G) > i > 0$ for some i . Let \mathcal{R} be a set whose elements are unions of i -islands, and let v be a vertex. Suppose that for each $R \in \mathcal{R}$, there exists a bridge structure B which interconnects $R \cup \{v\}$ and satisfies*

$$w_0 \#(B \cup \{v\} \setminus A_i) / |R| \leq \mathbf{i}(G) - i. \quad (3.4)$$

Then \mathcal{R} is finite.

PROOF. For $R \in \mathcal{R}$, let S denote $(B \cup \{v\}) \setminus A_i$, and let A denote the union of islands intersecting $B \cup R \cup \{v\}$. Then

$$|\partial(A \cup S)| \leq |\partial A| + |\partial S| \leq i|A| + |S|.$$

The bound on the first term of the second inequality holds since A is a union of islands. By (3.4) we have $|S|/|A| \leq \mathbf{i}(G) - i$. Therefore, using that A and S are disjoint, we get

$$\frac{|\partial(A \cup S)|}{|A \cup S|} \leq \frac{i|A| + |S|}{|A| + |S|} = \frac{i + |S|/|A|}{1 + |S|/|A|} \leq \frac{\mathbf{i}(G)}{1 + \mathbf{i}(G) - i} < \mathbf{i}(G).$$

By the anchored expansion property there are only finitely many such sets $A \cup S$ containing v . The lemma follows. \square

Proposition 3.6 *Let G be a graph with $\mathbf{i}(G) > 0$ and w_0 -bounded geometry. Let*

$$0 < i \leq \frac{2}{3}\mathbf{i}(G), \quad r(\ell) := a2^\ell/\ell^2, \quad a := \frac{3}{2}\pi^{-2}iw_0^{-1}.$$

*For each positive integer ℓ there is a (possibly empty) collection Ξ_ℓ of vertex sets C , which we call **level ℓ countries**, so that the following hold:*

- *For each ℓ and $C \in \Xi_\ell$, the set $C \cap A_i$ is a union of i -islands, and is called the **land** of the country C . Its volume satisfies $|C \cap A_i| \in [2^{\ell-1}, 2^\ell]$.*
- *For each ℓ and $C \in \Xi_\ell$, $C \setminus A_i = \{v \in V(G_i) : \text{dist}_i(v, C \cap A_i) \leq r(\ell)\}$, and this set is called the **waters** of the country C .*
- *Any two countries at the same level are disjoint.*
- *Every i -island is a subset of some country.*
- *Each vertex of G is contained in at most finitely many countries.*

PROOF. We start by constructing **regions** R , which are islands or unions of islands, together with bridge structures $B(R)$ connecting these islands if they are disjoint. First, for each $\ell \geq 1$, label each i -island with volume in $[2^{\ell-1}, 2^\ell]$ as a level ℓ region, and for these regions R , set $B(R) = \emptyset$.

Define the **waters** of a level ℓ region R as $\{v \in V(G_i) : \text{dist}_i(v, R) \leq r(\ell)\}$. Then, for $\ell = 1, 2, \dots$ (in this order), consider a maximal matching of pairs of level ℓ regions whose waters intersect, and label the union R of each matched pair (R_1, R_2) of regions a level $\ell + 1$ region. Set $B(R)$ to be the union of $B(R_1)$, $B(R_2)$ and a shortest (minimal length) bridge connecting R_1 and R_2 .

For every level ℓ region R and vertex v in R or in its waters, consider the bridge structure $B(v, R)$ given by the union of $B(R)$ and a shortest bridge connecting $\{v\}$ and R . We have

$$\#(B(v, R) \cup \{v\} \setminus A_i) \leq r(\ell) + \sum_{n=1}^{\ell-1} 2^{\ell-1-n} \cdot 2r(n) < a2^\ell \sum_{n=1}^{\infty} n^{-2} = \frac{i}{w_0} 2^{\ell-2}.$$

In the second expression the first term is an upper bound on the length of a shortest bridge connecting $\{v\}$ and R plus 1. The first factor in the sum is an upper bound on the number of pairs of level n regions contained in R ; the second factor is an upper bound of the length of a shortest bridge connecting such a pair.

Since then $w_0\#(B(v, R) \cup \{v\} \setminus A_i)/|R| < i/2 \leq \mathbf{i}(G) - i$, it follows by Lemma 3.5 that each vertex is contained in only finitely many regions or their waters. Therefore, every island is contained in a maximal region, that is a region which is not contained in any other regions. Call the union of a maximal region and its waters a country of level of the maximal region. Call the region itself the land of the country. This construction clearly satisfies the properties claimed in the proposition. \square

4 Random walk and anchored expansion

Let $\{X_n\}$ be the random walk on a graph G with i -anchored expansion and w_0 -bounded geometry. Our strategy for the analysis of this walk will be to handle the time spent in the

oceans and in the islands separately. Let N_m be the m -th positive time when $\{X_n\}$ visits a vertex in $G \setminus A_i$. We have seen that $\{X_{N_m}\}$ is the random walk on the graph G_i , which has Cheeger constant at least i . First, we show that

$$\liminf |X_{N_m}|/m \geq \frac{|\log(1-i^2)|}{\log w_0} > \frac{i^2}{\log w_0} \quad \text{a.s.} \quad (4.5)$$

For this, we first quote a version of the classical result of Cheeger (1970), Dodziuk (1984) and Mohar (1988), to be found, for example, in Lyons and Peres (1998).

Proposition 4.1 *Let P be the Markov kernel of the random walk on a weighted graph with Cheeger constant at least i . Then $\|P\| \leq (1-i^2)^{1/2} \leq (1-i^2/2)$.*

This, together with the following lemma implies (4.5).

Lemma 4.2 *Let G be a weighted graph with $\|P\| < 1$, and let f be a nonnegative vertex function so that*

$$g := \limsup |f^{-1}([0, n])|^{1/n} < \infty.$$

Then $\liminf f(X_n)/n \geq -2 \log \|P\| / \log g$ a.s.

If we set $f(v) := |v|$ in G (this might be different from $|v|$ measured in G_i), then the bounded geometry property implies that $g \leq w_0$, and the lemma applied to the walk X_{N_m} on G_i implies (4.5). In a similar fashion, we get the bound $i^2/\log g$ for the \liminf speed of random walks in graphs with Cheeger constant at least i and exponential growth rate at most g .

PROOF. For a small $\varepsilon > 0$, let $a := -2 \log \|P\| / \log(g + \varepsilon) - \varepsilon$. We have

$$\begin{aligned} \mathbf{P}_x[f(X_n) \leq an] &= \delta_x P^n \mathbf{1}_{f^{-1}([0, an])} = w(x)^{-1} (\mathbf{1}_x, P^n \mathbf{1}_{f^{-1}([0, an])}) \\ &\leq w(x)^{-1} \|P^n\| \|\mathbf{1}_{f^{-1}([0, an])}\| \end{aligned}$$

For sufficiently large n , this is bounded above by $w(x)^{-1} \|P\|^n (g + \varepsilon)^{an/2}$, which is summable, so $f(X_n) > an$ eventually a.s. \square

Our next goal is to bound the time spent in vertex sets, in particular, islands.

Lemma 4.3 *Let G be a graph with i -anchored expansion, let S be a vertex set, and suppose $i' \leq i$ is a constant so that S is contained in $G_{i'}$. Let n be an integer, $x \in V \setminus A_i$ be a vertex with $\text{dist}_i(x, S) \geq n + 1$, and let T be the time the random walk on G spends in S . Then we have*

$$\begin{aligned} \mathbf{P}_x(T > 0) &\leq 2w(x)^{-\frac{1}{2}} (1-i^2)^{\frac{n}{2}} i^{-2} |\partial S|^{\frac{1}{2}}, \\ \mathbf{E}_x T &\leq 2w(x)^{-\frac{1}{2}} (1-i^2)^{\frac{n}{2}} i'^{-2} |S|^{\frac{1}{2}}, \\ \mathbf{E}_x T^2 &\leq 8w(x)^{-\frac{1}{2}} (1-i^2)^{\frac{n}{2}} i'^{-4} |S|^{\frac{1}{2}}. \end{aligned}$$

For an arbitrary vertex x , these bounds hold with $n = 0$. If all edge weights are at least 1 and S is a union of islands, then we can use $i' := |S|^{-1}$.

PROOF. The quantities T , $w(x)$, $|\partial S|$, $|S|$ do not change if they are considered (for the walk) in the graph $G_{i'}$ instead of the graph G , so we will do this.

Denote P_i , \mathcal{G}_i , $P_{i'}$, $\mathcal{G}_{i'}$, the Markov and Green kernels of the walks on G_i , and $G_{i'}$, respectively. Recall that $\mathcal{G}_i = \sum_{m=0}^{\infty} P_i^m$, so we have

$$\|\mathcal{G}_i\| \leq \sum_{m=0}^{\infty} \|P_i\|^m = \frac{1}{1 - \|P_i\|},$$

and so from Proposition 4.1 we get

$$\|P_i\| \leq (1 - i^2)^{\frac{1}{2}} \leq (1 - i^2/2), \quad \|\mathcal{G}_i\| \leq 2i^{-2}, \quad (4.6)$$

and these inequalities also hold with i replaced by i' everywhere. For the walk on G_i started at $y \in V \setminus A_i$, the probability of moving into S from the outside in one step is given by the function $f(y)$ which equals $\delta_y P_{i'} \mathbf{1}_S$ outside S , and 0 in S . Thus the chance of moving into S from the outside after m steps in $V \setminus A_i$ is given by $\delta_y P_i^m f$, and therefore

$$\mathbf{P}_x(T > 0) \leq \sum_{m=n}^{\infty} \delta_x P_i^m f.$$

In Green kernel notation, this can be written as an inner product

$$w(x)^{-1}(\mathbf{1}_x, P_i^n \mathcal{G}_i f) \leq w(x)^{-1} \|\mathbf{1}_x\| \cdot \|P_i\|^n \cdot \|\mathcal{G}_i\| \cdot \|f\|.$$

The norms are all $L^2(V \setminus A_i, w)$, and the last inequality follows from the Schwarz inequality and the norm bounds. Since $f \leq 1$, the last norm is bounded above by $(f, 1)^{\frac{1}{2}}$, which equals $|\partial S|^{\frac{1}{2}}$. The first claim of the lemma now follows from (4.6).

For the expected value, write

$$\mathbf{E}_x T = \sum_{m=0}^{\infty} \delta_x P_i^m P_{i'}^m \mathbf{1}_S = w(x)^{-1}(\mathbf{1}_x, P_i^n \mathcal{G}_{i'} \mathbf{1}_S).$$

Since $\|\mathbf{1}_S\| = |S|^{\frac{1}{2}}$, the norm bound on the last formula and (4.6) give the second claim of the lemma.

Finally, denote $\{X_n\}$ the random walk on $G_{i'}$. Then

$$\mathbf{E}_x T^2 = \mathbf{E}_x \sum_{\substack{s, t > n \\ y, z \in S}} \mathbf{1}(X_s = y) \mathbf{1}(X_t = z),$$

and summing twice on or under the diagonal and extending the range of y gives the upper bound

$$2\mathbf{E}_x \sum_{\substack{s > n \\ d \geq 0}} \sum_{y \in V} \mathbf{1}(X_s = y) \mathbf{1}(X_{s+d} \in S).$$

By the Markov property this equals

$$2 \sum_{\substack{s > n \\ d \geq 0}} \sum_{y \in V} \mathbf{P}_x(X_s = y) \mathbf{P}_y(X_d \in S) = 2 \cdot \delta_x P_i^n \mathcal{G}_{i'} \mathcal{G}_{i'} \mathbf{1}_S.$$

The third claim of the lemma follows if we write this as an inner product and use norm bounds, as before. Omitting the estimates for the first n steps gives the proof for general x . Lemma 3.4 implies that we can use $i' := |S|^{-1}$. \square

The anchored expansion property suggests that large islands cannot be very frequent. The following lemma proves such a statement from the point of view of the random walk. It uses the hypotheses and the resulting decomposition of Proposition 3.6.

Lemma 4.4 *Consider countries C whose land is visited by time N_m , and let M_m be the volume of the largest such land. Then we have*

$$\limsup \frac{M_m}{\log m (\log \log m)^2} < c < \infty \quad a.s.$$

PROOF. For a positive b , let $g(n) := b \log n (\log \log n)^2$, and let \mathcal{A}_ℓ be the event that the land of a country of level ℓ is visited by time $g^{-1}(2^\ell)$. It suffices to prove that only finitely many of these events happen, which will follow if $\mathbf{P}\mathcal{A}_\ell \leq 2^{-\ell}$ for every large ℓ . Consider ℓ so large that the starting point of the walk is not contained in any level ℓ country, and $r(\ell) \geq 1$. Let T_C denote the first hitting time of a country C . Let $\mathcal{A}_{\ell,C}$ denote the event that the land of the country C is visited by time $g^{-1}(2^\ell)$, and let \mathcal{A}_C denote the event that the land of the country C is ever visited. The event $\mathcal{A}_{\ell,C}$ implies \mathcal{A}_C and $T_C \leq g^{-1}(2^\ell)$, and therefore

$$\mathbf{P}\mathcal{A}_{\ell,C} \leq \sum_{t=1}^{g^{-1}(2^\ell)} \mathbf{P}(\mathcal{A}_C | T_C = t) \mathbf{P}(T_C = t).$$

Summing over level ℓ countries we get

$$\begin{aligned} \mathbf{P}\mathcal{A}_\ell &\leq \sum_{C \in \Xi_\ell} \sum_{t=1}^{g^{-1}(2^\ell)} \mathbf{P}(\mathcal{A}_C | T_C = t) \mathbf{P}(T_C = t) \\ &\leq \sup_{\substack{C \in \Xi_\ell \\ 1 \leq t \leq g^{-1}(2^\ell)}} \mathbf{P}(\mathcal{A}_C | T_C = t) \sum_{t=1}^{g^{-1}(2^\ell)} \sum_{C \in \Xi_\ell} \mathbf{P}(T_C = t). \end{aligned} \quad (4.7)$$

For fixed t , the events in the inner summand of (4.7) are disjoint, so the second factor is bounded above by $g^{-1}(2^\ell)$. If C is a level ℓ country with land S , then

$$\text{dist}_i(X_{T_C}, S) = \lfloor r(\ell) \rfloor.$$

Therefore by the Strong Markov Property and Lemma 4.3, $\mathbf{P}(\mathcal{A}_C | T_C = t)$ is not more than

$$2w(x)^{-\frac{1}{2}} (1 - i^2)^{(r(\ell)-2)/2} i^{-2} |\partial S|^{\frac{1}{2}} \leq c' \exp(-c'_1 2^\ell / \ell^2) 2^{\ell/2} \leq c 2^{-\ell} \exp(-c_1 2^\ell / \ell^2).$$

Then by (4.7), $\mathbf{P}\mathcal{A}_\ell \leq g^{-1}(2^\ell) c 2^{-\ell} \exp(-c_1 2^\ell / \ell^2)$ and it suffices to prove that

$$g^{-1}(2^\ell) \leq c^{-1} \exp(c_1 2^\ell / \ell^2).$$

We apply g to both sides and use its monotonicity to transform the above to

$$2^\ell \leq b(c + c_1 \frac{2^\ell}{\ell^2} (\ell \log 2 - 2 \log \ell)^2).$$

This certainly holds for all large ℓ if b is large. □

The following corollary will be used in a later section. It implies that from the point of view of speed, distance can be measured while walking on water.

Corollary 4.5 *Set*

$$H_m := \inf_{N_{m-1} < n \leq N_m} |X_n|. \quad (4.8)$$

Then we have $\lim(H_m/|X_{N_m}|) = 1$ a.s.

PROOF. Between times N_{m-1} and N_m the walker is on an island with diameter bounded above by the volume M_m of the largest land visited by time N_m . Thus we have $|X_{N_m}| - M_m - 1 \leq H_m \leq |X_{N_m}|$. Dividing by $|X_{N_m}|$, and using the lemma together with (4.5) proves the corollary. \square

5 Lower bound on the speed

This section contains the proof of Theorem 1.1. We also give some counterexamples indicating why the bounded geometry condition is important.

Theorem 1.1 *There exists $c > 0$ so that the random walk on a weighted graph with w_0 -bounded geometry satisfies $\liminf |X_n|/n \geq c \mathbf{i}(G)^7 w_0^{-3}$ a.s.*

PROOF. Let G be a graph with the anchored expansion property and w_0 -bounded geometry, and consider the construction of countries from Proposition 3.6. Using the notation of the previous section, we can decompose the inverse \liminf speed \underline{S}^{-1} as

$$\begin{aligned} \limsup n/|X_n| &= \limsup_m \sup_{N_{m-1} < n \leq N_m} (n/|X_n|) \\ &\leq \limsup_m (N_m/H_m) = \limsup_m (N_m/|X_{N_m}|). \end{aligned} \quad (5.9)$$

H_m in the above expression is defined in (4.8), and the last equality follows from Corollary 4.5. Let $K_m := N_m - m$ denote the time spent in the islands up to time N_m . By (5.9) we have

$$\underline{S}^{-1} \leq \limsup (m/|X_{N_m}|)(1 + \limsup (K_m/m)).$$

The first factor in the last expression is the inverse of the \liminf speed in the graph G_i , for which we have the bound (4.5). Thus in order to show that \underline{S} is greater than a constant a.s. it suffices to find constants b_ℓ so that

$$\limsup (K_m/m) = \limsup (K_{m^2}/m^2) \leq \sum_{\ell \geq 1} b_\ell < \infty \quad \text{a.s.} \quad (5.10)$$

The equality holds since K_m is non-decreasing.

For each ℓ , if $X_0 = v$ is contained in a level ℓ country C , then set $C_{\ell,0} := C$, otherwise set $C_{\ell,0} := \emptyset$. Set $\tau_{\ell,0} := 0$, and for $k \geq 1$ define

$$\tau_{\ell,k} = \min\{n \geq \tau_{\ell,k-1} + 1 : X_n \in C =: C_{\ell,k} \text{ for some } C \in \Xi_\ell \setminus \{C_{\ell,k-1}\}\}.$$

Also, for $k \geq 0$, define the time spent in the land between stopping times:

$$T_{\ell,k} = \#\{n : \tau_{\ell,k} \leq n < \tau_{\ell,k+1}, X_n \in C_{\ell,k} \cap A_i\}.$$

We will use the rough bound $K_m \leq \sum_{\ell=1}^{\infty} \sum_{k=0}^m T_{\ell,k}$. Since each vertex is contained in at most finitely many countries, we have $T_{\ell,0} = 0$ for all but finitely many ℓ . So for (5.10) it suffices to find summable b_ℓ such that

$$\sum_{m \geq 1, \ell \geq 0} \mathbf{P}(\sum_{k=1}^{m^2} T_{\ell,k} > b_\ell m^2) < \infty. \quad (5.11)$$

Now fix ℓ , and suppose that $X_0 = v$ is on the inner vertex boundary of a level ℓ country with land R . If $r(\ell) \geq 1$, then this means that v is in the ocean and $\text{dist}_i(v, R) = \lfloor r(\ell) \rfloor$. Lemma 4.3 with $i' := |R|^{-1}$ gives

$$\mathbf{E}T_{\ell,0}^2 \leq 8(1 - i^2)^{(r(\ell)-2)/2} |R|^{4.5} \leq 8(1 - i^2)^{a_{2\ell-1}/\ell^2-1} 2^{4.5\ell} =: a_\ell^2. \quad (5.12)$$

If $r(\ell) < 1$, then v is contained in the land R , and this bound still holds (although it is very rough) by Lemma 4.3 applied to a general starting point. By the Markov property, this implies that for all $k \geq 1$, we have $\mathbf{E}(T_{\ell,k}^2 | \mathcal{F}(\tau_{\ell,k})) \leq a_\ell^2$, where $\mathcal{F}(\tau_{\ell,k})$ denotes the standard σ -field at the stopping time $\tau_{\ell,k}$, that is the σ -field generated by information available up to time $\tau_{\ell,k}$. Define

$$S_{\ell,m} := \sum_{k=1}^m (T_{\ell,k} - \mathbf{E}(T_{\ell,k} | \mathcal{F}(\tau_{\ell,k}))) \geq \sum_{k=1}^m (T_{\ell,k} - a_\ell).$$

Since $\{S_{\ell,m}\}_{m \geq 1}$ is a martingale, we can write

$$\text{Var } S_{\ell,m} = \sum_{k=1}^m \mathbf{E} \text{Var}(T_{\ell,k} | \mathcal{F}(\tau_{\ell,k})) \leq m a_\ell^2.$$

Therefore, if $b_\ell > a_\ell$, then Chebyshev's inequality gives

$$\mathbf{P}(\sum_{k=1}^m T_{\ell,k} > m b_\ell) \leq \mathbf{P}(S_{\ell,m} > m(b_\ell - a_\ell)) \leq \frac{m a_\ell^2}{m^2 (b_\ell - a_\ell)^2}.$$

Thus if we set, for example, $b_\ell := (\ell + 1)a_\ell$, then it is clear from looking at the expression of a_ℓ that the conditions of (5.11) and (5.10) are satisfied. We thus have proved that the speed is greater than a constant depending on i and w_0 only.

It remains to give a bound on the constant in terms of i and w_0 . Since b_ℓ can be large when ℓ is small, in order to get a reasonable bound, we need to deal with countries at or below some minimal level ℓ_0 separately.

The value of ℓ_0 will be determined later, for now just assume that $2^{-\ell_0} \leq i$. Then by Lemma 3.4 the land of countries of level up to ℓ_0 is contained in $V \setminus A_{2^{-\ell_0}}$. Let K_m^* denote the time the random walk spends in $V \setminus A_{2^{-\ell_0}}$ by time N_m , and let K'_m denote the time the walk spends in the land of countries of level greater than ℓ_0 by time N_m . We then have $N_m \leq K_m^* + K'_m$.

Note that the sequence $\{K_m^*/|X_{N_m}|\}$ is a subsequence of $\{m/|X_{N'_m}|\}$, where N'_m is the time of the m th visit to $G_{2^{-\ell_0}}$. By (4.5), the lim sup of this sequence, and thus the lim sup of the first one, is at most $(2^{\ell_0})^2 \log w_0$. The bound (4.5) on $\limsup(m/|X_{N_m}|)$, (5.9), and the bound (5.10) on $\limsup(K'_\ell/m)$ from the first part of the proof imply

$$\begin{aligned} \underline{S}^{-1} &\leq \limsup(K_m^*/|X_{N_m}|) + \limsup(m/|X_{N_m}|) \limsup(K'_m/m) \\ &\leq (2^{\ell_0})^2 \log w_0 + i^{-2} \log w_0 \sum_{\ell > \ell_0} b_\ell. \end{aligned} \quad (5.13)$$

We now want to choose an ℓ_0 so that the last sum is small, say each term b_ℓ is at most $2^{-\ell}$. From (5.12), since $\ell \geq 1$, we have

$$2^\ell b_\ell \leq \exp(c\ell - c_2 \alpha^{-1} 2^\ell / \ell^2) \quad (5.14)$$

with $\alpha := w_0 i^{-3} \vee 2$. A simple computation shows that there is a constant $c_3 = c_3(c, c_2) \geq 1$ so that the right hand side of (5.14) is at most 1 if $\ell \geq \ell_0$, where ℓ_0 is chosen so that

$$2^{\ell_0} = c_3 \alpha (\log_2 \alpha)^3.$$

Using this choice of ℓ_0 , from (5.13) we conclude that

$$\underline{S}^{-1} \leq c_3^2 \alpha^2 (\log_2 \alpha)^6 \log w_0 + i^{-2} \log w_0 \leq c^{-1} w_0^3 \mathbf{i}(G)^{-7}. \quad \square$$

Example 5.1 Consider the binary tree with edge weights 1, and for each n attach an extra vertex to each vertex at distance n from the root by an edge with weight $1/(n \log n)$. Add a self-loop to each new vertex so that it will have weight 1. This graph clearly has anchored expansion. The walk will visit infinitely many of these new vertices by the Borel-Cantelli Lemma, and at each visit it has at least constant probability to spend time at least $n \log n$ at the vertex. Thus in this graph the speed is zero; this shows that in Theorem 1.1 the bounded geometry condition cannot be left out, nor replaced by bounds on the vertex weights.

Example 5.2 It follows from bounded geometry that positive transition probabilities are bounded from below. This is another weaker condition, but too weak for Theorem 1.1. Define the **pipe** of length n as the nearest neighbor graph on $0, 1, 2, \dots, n$. Consider the binary tree, and for every n and vertex at distance n , add a pipe of length $2k$ with edge weights $1, 2^{-1}, \dots, 2^{-k+1}, 2^{-k}, 2^{-k+1}, \dots, 1$ with $2^k \approx n \log n$. The argument of the previous example applies again.

6 A heat kernel bound

This section contains the proof of Theorem 1.2 and examples showing that the bound there is sharp up to the constant factor in the exponent.

Theorem 1.2 *Let G be a weighted graph with the anchored expansion property and w_0 -bounded geometry. Let $\alpha := \mathbf{i}(G)^2 (w_0^2/2)^{-1/3}/9$. For every vertex x there is an N so that*

$$p^n(x, y) < e^{-\alpha n^{1/3}} \quad \text{for all } n > N, y \in V.$$

PROOF. Fix a vertex x , and let $a_n := an^{1/3}$ for $a > 0$ to be determined later. Let $i := \frac{2}{3}\mathbf{i}(G)$, let $A_{i,n} \subset A_i$ be the union of islands with volume at least a_n , and define the **territory** of such an island C as the set of vertices $v \in V$ with

$$\text{dist}_i(v, C) \leq a_n i / (4w_0). \quad (6.15)$$

Lemma 3.5 implies that there are only finitely many n for which x is contained in the territory of an island of $A_{i,n}$. Consider large n for which (i) this does not happen and (ii) the right hand side of (6.15) is at least 1. Condition (ii) and the definition of dist_i ensures that the inner vertex boundary of the territory of an island is a subset of the ocean, $V \setminus A_i$. Condition (i) implies that $1/a_n < i$, and, as shown in Section 3, $A_{1/a_n} \subset A_i$. By Lemma 3.4, islands of A_i with volume less than a_n do not intersect A_{1/a_n} , so $A_{1/a_n} \subset A_{i,n}$, and we have $x \in V(G_{1/a_n})$.

Let $A'_{i,n}$ denote the union of islands of $A_{i,n}$ which are at distance at most n from x , and let $p' = p'(n)$ denote the transition kernel of the walk on G_{1/a_n} . Note that $p^n(x, y)$ is the sum of the probabilities of paths of length n starting at x and ending at y . Each such path stays in G_{1/a_n} or visits $A'_{i,n}$. The total probability of the first kind of paths is at most $p'^n(x, y)$ (regarded as 0 if $y \in A_{1/a_n}$), so for all y we have

$$p^n(x, y) \leq p'^n(x, y) + \mathbf{P}_x(\{X_k\} \text{ hits } A'_{i,n}).$$

The first term on the right satisfies

$$\begin{aligned} p'^n(x, y) &= w(x)^{-1}(\mathbf{1}_x, P_{1/a_n}^n \mathbf{1}_y) \leq w_0^{\frac{1}{2}} \|P_{1/a_n}\|^n \\ &\leq w_0^{\frac{1}{2}} (1 - a_n^{-2})^{\frac{n}{2}} < w_0^{\frac{1}{2}} \exp(-\frac{1}{2}na_n^{-2}). \end{aligned} \quad (6.16)$$

The first inequality follows from Cauchy-Schwarz, the second from Proposition 4.1 and the fact that G_{1/a_n} has Cheeger constant at least $1/a_n$.

Suppose that there is a union R_n of $n+1$ islands in $A'_{i,n}$ so that the territory of the first intersects the territory of all the others. Then there is bridge structure B interconnecting $R_n \cup \{x\}$ with

$$\#(B \cup \{x\} \setminus A_i) \leq n + n(\frac{i}{2}w_0^{-1}a_n - 1) \leq w_0^{-1}\frac{i}{2}|R|.$$

In the second expression, second term in parentheses is an upper bound on the number of vertices in a bridge connecting two islands with intersecting territories, and the first n is an upper bound for the number of vertices needed for the connection to x . Lemma 3.5 then implies that there are finitely many n such that R_n exists. Thus for all large n , it is possible to n -color islands in $A'_{i,n}$ so that islands of the same color have disjoint territories. For such n , the probability of hitting some island is bounded by n times the maximal probability of hitting some island of a given color.

Suppose that the walk starts at a vertex v on the inner vertex boundary of the territory of an island $C \subset A'_{i,n}$. By construction, this means that

$$\text{dist}_i(v, C) = \lfloor a_n i / (4w_0) \rfloor \geq 1.$$

Also note that another application of Lemma 3.5 shows that for some c and all large n , $A'_{i,n}$ cannot contain islands with volume larger than cn . For such n , by Lemma 4.3 the probability of hitting C is bounded by

$$2w(x)^{-\frac{1}{2}}(1 - i^2)^{(a_n i / (4w_0) - 2)/2} i^{-2} |\partial C|^{\frac{1}{2}} \leq c(i, w_0) n^{1/2} (1 - i^2)^{(a_n i / (4w_0) - 2)/2}.$$

In the first n steps the walker has at most n occasions to be at the inner vertex boundary of some island of a given color. Thus by the Markov property we get the bound

$$\mathbf{P}_x(\{X_k\} \text{ hits } A'_{i,n}) \leq n \cdot n \cdot cn^{1/2} \exp\left(\log(1 - i^2)ia_n/(8w_0)\right). \quad (6.17)$$

There exists an $a < (2\alpha)^{-1/2}$ so that the exponents of (6.16) and (6.17) are at most $-cn^{-1/3}$ with $c > \alpha$. The statement of the theorem follows. \square

Example 6.1 Let $\{X_n\}$ be the nearest neighbor walk on the nonnegative integers started at 0. There is a constant a so that $\mathbf{P}(X_1, \dots, X_{n^3} < n) > e^{-an}$ for all n . Consider the binary tree with pipes of length ℓ_n attached to a vertex v_n at distance ℓ_n from the root o for some rapidly increasing sequence $\{\ell_n\}$. This graph has anchored expansion. However, consider the set of possible paths of length ℓ_n^3 which start and end at o . A subset of these start at o , travel on a shortest path to the opposite end of the pipe starting at v_n , spend time $\ell_n^3 - 4\ell_n$ in the pipe, and use the remaining time to return to o . By the above, the probability measure of this set of paths is at least $(1/4)^{4\ell_n} e^{-a\ell_n}$, and we get $p^{\ell_n^3}(o, o) > e^{-c\ell_n}$. This shows that the conclusion of Theorem 1.2 is sharp up to the constant in the exponent.

Example 6.2 Chen and Peres (1998) showed that a supercritical Galton-Watson tree has anchored expansion, so the above theorem gives the $e^{-cn^{1/3}}$ heat kernel upper bound. In this case, such bounds are immediate from results of Piau. For Galton-Watson trees where the probability of non-branching (zero or one offspring) is positive this bound is easily seen to be sharp up to the constant in the exponent (see Piau 1998).

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